The order notation

If you studied series at school you will have seen series such as:

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3 + \cdots$$

This series goes on for ever, and in some contexts it is important to know how the 'remainder' terms, the bits contained in the '$\cdots$' behave. One way to be a little more accurate would be to write this series as:

$$e^x = 1 + x + \tfrac{1}{2}x^2 + \text{terms in } x^3 \ \textit{etc}$$

What we are thinking here is that, if x is a fairly small number (say 0.1) then the term in x^3 will be much smaller than the earlier terms, and the terms in even higher powers will be smaller still.

In algebraic calculations where it is important to be clear about the behaviour of the remainder term, we can use a special notation where we would write:

$$e^x = 1 + x + \tfrac{1}{2}x^2 + O(x^3) \text{ as } x \to 0$$

or:

$$e^x = 1 + x + \tfrac{1}{2}x^2 + o(x^2) \text{ as } x \to 0$$

Note that the first equation uses a capital 'O' while the second uses a small 'o'.

O() Notation

The $O(x^3)$ in the first equation is read as 'Big-Oh of *x* cubed'. You can think of this as representing some function that is no bigger than a fixed multiple of x^3. Mathematically speaking, a function $f(x)$ is $O(x^3)$ if you can find **at least one constant** $K > 0$ such that $|f(x)| \le K|x^3|$ for all sufficiently small values of x.

Another way of writing this would be to say that $\left|\dfrac{f(x)}{x^3}\right| \le K$, *ie* $f(x)/x^3$ is bounded by some constant K for sufficiently small values of x.

o() Notation

The $o(x^3)$ in the second equation is read as 'Small-Oh of *x* cubed'. You can think of this as representing some function that is smaller than any fixed multiple of x^3. Mathematically speaking, a function $f(x)$ is $o(x^3)$ if **for all constants** $K>0$, $|f(x)|\leq K|x^3|$ for all sufficiently small values of x.

Another way of writing this would be to say that $\left|\frac{f(x)}{x^3}\right|\leq K$, *ie* $f(x)/x^3$ is bounded by **all** positive constants K for sufficiently small values of x, which means that $\lim_{x\to 0}\left|\frac{f(x)}{x^3}\right|=0$.

So small o is a stronger condition than big O.

Note that if you're consistently using big O's or small o's in a derivation you can just say 'order of *x* cubed' *etc*.

Example

Which of the following functions are $O(x^3)$ and/or $o(x^3)$ for small values of x?

(i) $5x^2$

(ii) $0.00001x^3$

(iii) $999{,}999{,}999x^4$

(iv) $\frac{e^x-e^{-x}}{x}$

Solution

(i) $\left|\frac{5x^2}{x^3}\right|=\frac{5}{|x|}$ so as x gets smaller $5/x$ gets larger so it will not be bounded by some constant M, so it's not $O(x^3)$. It's also not heading to zero, so it's not $o(x^3)$.

(ii) $\left|\dfrac{0.00001x^3}{x^3}\right| = 0.00001$ so it's clearly bounded by $M = 0.00001$ (or any bigger number) and hence it is $O(x^3)$. But it doesn't approach zero, so it's not $o(x^3)$.

Notice that the 0.00001 factor doesn't affect the orders here.

(iii) $\left|\dfrac{999{,}999{,}999x^4}{x^3}\right| = 999{,}999{,}999x$ so it will be bounded for small values of x (for example it will be bounded by 999,999,999 for values of x smaller than 1) and hence it is $O(x^3)$. It also heads to zero as $x \to 0$ so it is also $o(x^3)$.

Again the 999,999,999 factor doesn't make any difference.

Note that being $o(x^3)$ is a stronger condition than being $O(x^3)$. Any function that is $o(x^3)$ is automatically $O(x^3)$.

(iv) For $\dfrac{e^x - e^{-x}}{x}$, we have to do a 'calculation' involving the order symbols. We know that:

$$e^x = 1 + x + \tfrac{1}{2}x^2 + O(x^3)$$

and:

$$e^{-x} = 1 - x + \tfrac{1}{2}x^2 + O(x^3)$$

Notice that we've written $+O(x^3)$ rather than $-O(x^3)$ in the second equation. We can do this because these two symbols are equivalent.

If we now evaluate the numerator, we get:

$$e^x - e^{-x} = \left[1 + x + \tfrac{1}{2}x^2 + O(x^3)\right] - \left[1 - x + \tfrac{1}{2}x^2 + O(x^3)\right]$$

Simplifying:

$$e^x - e^{-x} = 2x + O(x^3)$$

Again notice that adding two $O(x^3)$ terms just gives $O(x^3)$.

We now need to divide by x:

$$\frac{e^x - e^{-x}}{x} = 2 + \frac{O(x^3)}{x}$$

This last term is a function that is no bigger than a fixed multiple of x^3 divided by x. This will give a function that is no bigger than a fixed multiple of x^2 *ie* $O(x^2)$.

So:

$$\frac{e^x - e^{-x}}{x} = 2 + O(x^2)$$

Since $O(x^2)$ is 'bigger' than $O(x^3)$, we can conclude that $\frac{e^x - e^{-x}}{x}$ is neither $O(x^3)$ nor $o(x^3)$.

You may also see the order notation used to describe the behaviour of a function for large values of x.

Example

Explain what the statement $\frac{x}{x-1}=1+\frac{1}{x}+O(x^{-2})$ as $x\to+\infty$ means.

Solution

This means that the behaviour of the function $\frac{x}{x-1}$ is very similar to the behaviour of the function $1+\frac{1}{x}$ for large values of *x*. The remainder (discrepancy) is a term that approaches zero more quickly than any fixed multiple of $\frac{1}{x^2}$.

You can see this if you expand $\frac{x}{x-1}$ by writing it as:

$$\frac{x}{x-1}=\frac{1}{1-\frac{1}{x}}=\left(1-\tfrac{1}{x}\right)^{-1}=1+\frac{1}{x}+\frac{1}{x^2}+\frac{1}{x^3}+\cdots$$

Supremums and infimums

For finite sets of numbers, you can always identify which element(s) is/are the largest/smallest. For infinite sets, this is not always possible, and we have to generalise the idea of maximum/minimum.

The supremum (sup), or least upper bound of a set *A* is defined as the number α such that $\alpha\geq a\ \forall\ a\in A$ and $\forall\ a<\alpha\ \exists\ a'>a$.

In other words $\sup(A)$ is the number that is 'never quite' exceeded by any member of the set. It is literally the 'least upper bound'.

For example, consider the sequence $\tfrac{1}{2},\ \tfrac{2}{3},\ \tfrac{3}{4},\ \ldots$. This sequence has no maximum, but the supremum of the sequence is 1, since any number less than 1 will eventually be exceeded and any other upper bound would be greater than 1.

Similarly the idea of the minimum element is generalised to give the infimum or greatest lower bound.

The infimum (inf), or greatest lower bound of a set *A* is defined as the number α such that $\alpha \le a \ \forall \ a \in A$ and $\forall \ a > \alpha \ \exists \ a' < a$.

Note that any other lower bound α' is such that $\alpha' \le \alpha$. Hence α is the greatest lower bound.

For example, consider the sequence $\frac{1}{2}, \ \frac{1}{4}, \ \frac{1}{8}, \ \ldots$. This sequence has no minimum, but the infimum of the sequence is 0.

Note that for finite sets *A*, $\sup(A) = \max_{a \in A}(A)$ and $\inf(A) = \min_{a \in A}(A)$.